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On perturbing Liapunov functional

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Abstract

Perturbing Liapunov function method was introduced for systems of ordinary differential equations. In this paper, we will extend this method to systems of functional differential equations and discuss stability and boundedness properties via the concept of perturbing Liapunov function.
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1. Introduction

Stability properties of systems of differential equations have been interesting important from the views of many authors (see [2,3]). Liapunov function has played an essential rule for determining the qualitative properties of the zero solution of the systems of differential equations. In [6], Lakshmikantham and Leela introduced the method of perturbing Liapunov function which discussed nonuniform properties of solutions of differential equations.

The main purpose of this paper is to extend this idea to systems of functional differential equations

$$x' = f(t, x_t), \quad x_{t_0} = \psi, \tag{1.1}$$

where $f \in C[J \times C_0, \Re^n], \Re^n$ is the Euclidean n-dimensional real space, $J = [t_0, \infty]$,

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$$\wp^n = C[[-r,0],\Re^n], \quad C_0 = \{\phi \in \wp^n: \|\phi\|_0 < \rho\}$$
$$\text{and } \|\phi\|_0 = \max_{-r \leqslant s \geqslant 0} \|\phi(s)\|,$$

$C[J \times C_0, \Re^n]$ denotes the space of continuous mapping $J \times C_0$ into $\Re^n$. For $x_t(s) = x(t+s)$, $-r \leqslant s \leqslant 0$ and $x_t(t_0, \psi)$ being a solution of (1.1) with initial values $x_{t_0} = \psi$. Define

$$s_0(\rho) = \{x_t \in C_0: \|x_t\| < \rho\}.$$

Following [5], we define for a Liapunov functional $V(t,x_t) \in C[J \times C_0, \Re^n]$ is Lipschitzian in x_t, the functional

$$D^+V(t,x_t) = \lim_{h \to 0^+} \sup \frac{1}{h}[V(t+h, x_{t+h}) - V(t,x_t)].$$

The first work dedicated to this method was done by Lakshmikantham and Leela [6].

The following definitions will be needed in the sequel.

Definition 1 [4]. A function $b(r)$ is said to belong the class $\aleph$ if $b \in C[[0,\rho),\Re^+]$, $b(0) = 0$, $b(r) \to 0$, as $r \to 0$, and $b(r)$ is strictly monotone increasing in r.

Definition 2 [5]. A solution $x_t(t_0,\psi)$ of (1.1) is said to be equibounded if there exist a positive constant $M(t_0,\delta) > 0$ and $\delta > 0$ such that for

$$\|\psi\| \leqslant \delta \Rightarrow \|x_t(t_0,\psi)\| \leqslant M(t_0,\delta).$$

Definition 3 [5]. The zero solution of (1.1) is said to be equistable if for $\epsilon > 0, t_0 \in J$ there exists a positive function $\delta(t_0,\epsilon) > 0$ that is continuous in t_0 such that

$$\|x_t(t_0,\psi)\| < \epsilon, \quad t \geqslant t_0.$$

provided that $\|\psi\|_0 < \delta$.

In the case of being uniformly stable δ is independent of t_0.

2. Equiboundedness

In this section, we discuss the boundedness of the systems (1.1) via perturbing Liapunov functional.

Theorem 1. *Let $E \subset C_0$ be compact subset. Suppose that there exist two functionals $V_1(t,x_t) \in C[J \times \overline{E}^c, \Re^+]$, and there exist two functions $V_2(t,x_t) \in C[J \times$*

$s_0^c(\rho), \Re^+]$, $g_1 \in C[\Re^+ \times \Re^+, \Re]$, *and* $g_2 \in C[\Re^+ \times \Re^+, \Re]$ *with* $V_1(t,0) = V_2(t, 0) = g_1(t,0) = g_2(t,0) = 0$ *such that*

(H1) $V_1(t,x_t)$ *is Lipschitzian in* x_t *and*

$$D^+V_1(t,x_t) \leqslant g_1(t, V_1(t,x_t)), \quad (t,x_t) \in J \times \overline{E}^c. \tag{2.1}$$

(H2) $V_2(t,x_t)$ *is Lipschitzian with respect to* x_t *and*

$$b\|x_t\| \leqslant V_2(t,x_t) \leqslant a\|x_t\|, \tag{2.2}$$

where $a, b \in \aleph$, $(t,x_t) \in (J \times s_0^c)$.

(H3) *For each* $(t,x_t) \in J \times s_0^c$,

$$D^+V_1(t,x_t) + D^+V_2(t,x_t) \leqslant g_2(t, V_1(t,x_t) + V_2(t,x_t)). \tag{2.3}$$

(H4) *If the zero solution of the scalar differential* equation

$$u' = g_1(t,u), \quad u(t,u) = u_0, \tag{2.4}$$

is equibounded, and if the zero solution of the scalar differential equation

$$w' = g_2(t,w), \quad w(t_0) = w_0, \tag{2.5}$$

is uniformly bounded.
Then the zero solution of (1.1) *is equibounded.*

Proof. Since E is compact subset of C_0, there exists $\rho > 0$ such that $s_0(\rho) \supset s_0(E, \rho_0)$ for some $\rho_0 > 0$, where

$$s_0(E,\rho_0) = \{x_t \in C_0 \colon d(x_t, E) < \rho_0\},$$

where

$$d(x_t, E) = \inf_{y_t \in E} \|x_t - y_t\|.$$

Let $t \in \Re^+$, $\alpha \leqslant \rho$ be given. Assume that $\alpha_1 = \alpha_1(t_0, \alpha) = \max(\alpha_0, \alpha^*)$, where

$$\alpha_0 = \max[V_1(t_0, \psi) \colon \psi \in \overline{s_0(\alpha) \cap E^c}]$$

and $a^* \geqslant V_1(t,x_t)$ for $(t,x_t) \in J \times \partial E$.

Since the zero solution of (2.4) is equibounded, given $\alpha_1 > 0$ and $t_0 \in \Re^+$, there exists $\beta_0 = \beta_0(t, \alpha_1)$ such that

$$u(t,t_0,u_0) < \beta_0, \quad t \geqslant t_0, \tag{2.6}$$

whenever $u_0 < \alpha_1$, where $u(t,t_0,u_0)$ is any solution of (2.4).

Also, since the zero solution of (2.5) is uniformly bounded, given $\alpha_2 > 0$, $t_0 \in \Re^+$, there exists $\beta_1(\alpha_2) > 0$ such that

$$w(t,t_0,u_0) < \beta_1(\alpha_1), \quad t \geqslant t_0, \tag{2.7}$$

provided that $w_0 < \alpha_2$, where $w(t, t_0, w_0)$ is any solution of (2.5).

Now, we choose $u_0 = V_1(t_0, \psi)$, and $\alpha_2 = a(\alpha) + \beta_0$, as $b(u) \to \infty$ with $u \to \infty$. We can choose $\beta = \beta(t_0, \alpha)$ such that

$$b(\beta) > \beta_1(\alpha_2). \tag{2.8}$$

Now, let $\psi \in s_0(\alpha)$ imply that any solution $x_t(t, \psi)$ satisfies $x_t(t, \psi) \in s_0(\beta)$ for $t \geqslant t_0$. Suppose that this is not true, there exists a solution $x_t(t_0, \psi)$ of (1.1) with $\psi \in s_0(\alpha)$ such that for $t^* > t_0$,

$$x_{t^*}(t_0, \psi) = \beta.$$

Since $s_0(E, \rho) \subset s_0(\alpha)$, there are two possibilities to consider
(I) $x_t(t_0, \psi) \in E^c$ for $t \in [t_0, t^*]$;
(II) there exists $t_2 \geqslant t_0$ such that

$$x_t(t_0, \psi) \in \partial E, \ \ x_t(t_0, \psi) \in E^c \quad \text{for } t \in [t_0, t^*].$$

If case (I) is true, then we can find $t_1 > t_0$ such that

$$x_{t_1}(t_0, \psi) = \alpha, \ \ x_{t^*}(t_0, \psi) = \beta, \ \text{ and } \ x_t(t_0, \psi) \in s_0^*(\alpha), \quad t \in [t_0, t^*] \tag{2.9}$$

setting

$$m(t) = V_1(t, x_t(t_0, \psi)) + V_2(t, x_t(t_0, \psi)), \quad t \in [t_1, t^*].$$

It is easy to obtain, from (2.3), thus from [1,5],

$$D^+ m(t) \leqslant g_2(t, m(t)), \quad t \in [t_1, t^*].$$

Consequently, by comparison [4, Theorem 1.4.1], we get

$$m(t) \leqslant r_2(t, t_1, m(t)), \quad t \in [t_1, t^*],$$

where $r_2(t, t_1, w_0)$ is the maximal solution of (2.5) such that $r_2(t_1, t_1, w_0) = w_0$. Thus

$$\begin{aligned} &V_1(t^*, x_{t^*}, (t_0, \psi)) + V_2(t^*, x_{t^*}(t_0, \psi)) \\ &\quad \leqslant r_2(t^*, t_1, V_1(t_1, x_{t_1}(t_0, \psi)) + V_2(t_1, x_{t_1}(t_0, \psi))). \end{aligned} \tag{2.10}$$

Similarly, from (2.1) we have

$$V_1(t_1, x_{t_1}(t_0, \psi)) \leqslant r_1(t_1, t_0, V_1(t_0, \psi)), \tag{2.11}$$

where $r_1(t_1, t_0, u_0)$ is the maximal solution of (2.4).

From the fact that $u_0 = V_1(t_0, \psi_1) < \alpha_1$, and (2.6) yield

$$r_1(t_1, t_0, V_1(t_0, \psi)) \leqslant \beta_0. \tag{2.12}$$

Furthermore,

$$V_2(t_1, x_{t_1}(t_0, \psi)) \leqslant a(\alpha), \tag{2.13}$$

from (2.12) and (2.13), we have

$$w_0 = V_1(t_1, x_{t_1}(t_0, \psi)) + V_2(t_1, x_{t_1}(t_0, \psi)) < \beta_0 + a(\alpha) = \alpha_2. \tag{2.14}$$

Hence, from (2.2), (2.7)–(2.10), (2.14), and the fact that $V_1 \geqslant 0$,

$$b(\beta) \leqslant \beta_1(\alpha_2) \leqslant b(\beta). \tag{2.15}$$

If the case (II) holds, we again arrive at the inequality (2.10), where $t_1 > t$ satisfies (2.9). Now, we have in place of (2.11) the inequality

$$V_1(t_1, x_{t_1}(t_0, \psi)) \leqslant r_1(t_1, t_2, V_1(t_2, x_{t_2}(t_0, \psi))).$$

Since $x_{t_2}(t_0, \psi) \in \partial E$ and $V_1(t_2, x_{t_2}(t_0, \psi)) \leqslant \alpha^* \leqslant \alpha_1$. Thus, we get the same contradiction in (2.15). This proves that

$$\|x_t(t_0, \psi)\| < \beta \quad \text{for } t \geqslant t_0,$$

whenever $\|\psi\| < \alpha,\ \alpha \geqslant \rho$. For $\alpha < \rho$, we set $\beta(t_0, \alpha) = \beta(t_0, \rho)$, and the proof is completed. □

3. Equistability

In this section, we discuss the concept of perturbing Liapunov functional for stability property of the system of functional differential equations (1.1).

Theorem 2. *Suppose that there exist two functionals g_1 and g_2 which are defined as in Theorem* 1, *and let there exist two functions $V_1(t, x_t) \in C[J \times s_0(\rho), \mathfrak{R}^+]$, $V_2(t, x_t) \in C[J \times s_0(\rho) \cap s_0^c(\eta), \mathfrak{R}^+]$, with $V_1(t,0) = V_2(t,0) = g_1(t,0) = g_2(t,0) = 0$ such that*

(H5) *$V_1(t, x_t)$ is Lipschitzian in x_t and*

$$D^+ V_1(t, x_t) \leqslant g_1(t, V_1(t, x_t)), \quad (t, x_t) \in J \times s_0(\rho). \tag{3.1}$$

(H6) *$V_2(t, x_t)$ is Lipschitzian in x_t and*

$$b\|x_t\| \leqslant V_2(t, x_t) \leqslant a\|x_t\|, \tag{3.2}$$

where $a, b \in \aleph, (t, x_t) \in (J \times S_0(\rho) \cap s_0^c(\eta))$.

(H7) *For each $(t, x_t) \in (J \times s_0(\rho) \cup s_0^c(\eta))$,*

$$D^+ V_1(t, x_t) + D^+ V_2(t, x_t) \leqslant g_2(t, V_1(t, x_t) + V_2(t, x_t)).$$

(H8) *If the zero solution of the scalar differential equation* (2.4) *is equistability, and if the zero solution of the scalar differential equation* (2.5) *is uniformly stability.*
Then the zero solution of (1.1) *is equistable.*

Proof. From our assumption, the zero solution of (2.5) is uniformly stable. Let $0 < \epsilon < \rho$ and $t_0 \in \Re$. Given $b(\epsilon) > 0$ and $t_0 \in \Re^+$, there exists $\delta_0 = \delta_0(\epsilon) > 0$ such that

$$w(t, t_0, w_0) < b(\epsilon), \quad t \geqslant t_0, \tag{3.3}$$

provided that $w_0 < \delta$, where $w(t, t_0, w_0)$ is any solution of (2.5).

From the condition (H6), there exists $\delta_2 = \delta_2(\epsilon) > 0$ such that

$$a(\delta_0) < \frac{\delta}{2}. \tag{3.4}$$

From our assumption, the zero solution of (2.4) is equistable, given $\delta/2$, and $t_0 \in \Re^+$, there exists $\delta^* = \delta^*(t_0, \epsilon) > 0$ such that

$$u(t, t_0, u_0) < \frac{\delta}{2}, \quad t \geqslant t_0, \tag{3.5}$$

provided that $u_0 < \delta^*$, $u(t, t_0, u_0)$ being any solution of (2.4).

Following [6], choose $u_0 = V_1(t_0, u_0)$, since $V_1(t, x_t)$ is continuous and $V_1(t, 0) = 0$, there exists $\delta_1 > 0$ such that

$$\|\psi\| < \delta \Rightarrow \|x_t(t_0, \psi)\| < \epsilon, \quad t \geqslant t_0. \tag{3.6}$$

Suppose that this is not true, there exist $t_1, t_2 > t_0$ such that for $\|\psi\| < \delta$,

$$\begin{aligned} &\|x_{t_1}(t_0, \psi)\| = \epsilon, \\ &\|x_{t_2}(t_0, \psi)\| = \delta, \\ &x_t(t_0, \psi) \in s_0(\epsilon) \cap s_0^c(\delta_0), \quad t \in [t_1, t_2]. \end{aligned} \tag{3.7}$$

Let $\delta_2 = \eta$, so that the condition (H6) is assured. Setting

$$m(t) = V_1(t, x_t(t_0, \psi)) + V_{2.\eta}(t, x_t(t_0, \psi)), \quad t \in [t_1, t_0].$$

we get

$$D^+ m(t) \leqslant g_2(t, m(t)), \quad t \in [t_1, t_2],$$

which yields

$$\begin{aligned} V_1(t_2, x_{t_2}(t_0, \psi)) + V_{2.\eta}(t_2, x_{t_2}(t_0, \psi)) \leqslant r_2(t_2, t_1, V_1(t_1, x_{t_1}(t_0, \psi)) \\ + V_{2.\eta}(t_1, x_{t_1}(t_0, \psi))), \end{aligned}$$

where $r_2(t_1, t_1, w_0) = w_0$, $r_2(t_1, t_1, w_0)$ is the maximal solution of (2.5).

Also, we have

$$V_1(t_1, x_{t_1}(t_0, \psi)) \leqslant r_1(t_1, t_0, V_1(t_0, \psi)),$$

where $r_1(t_1, t_0, u_0)$ is the maximal solution of (2.4).

By (3.5) and (3.6) we have

$$V_1(t_1, x_{t_1}(t_0, \psi)) \leqslant \frac{\delta}{2}. \tag{3.8}$$

From (3.4) and (3.7) we get

$$V_{2.\eta}(t_1, x_{t_1}(t_0, \psi)) < \frac{\delta}{2}. \tag{3.9}$$

Thus (3.3), (3.7)–(3.9), and (H6) yield the following contradiction:

$$\begin{aligned} b(\epsilon) &= b\|x_{t_1}(t_0, \psi)\| \\ &\leqslant V_{2.\eta}(t_1, x_{t_1}(t_0, \psi)) \\ &\leqslant a\|x_{t_2}(t_0, \psi)\| \\ &= a(\delta) \leqslant b(\epsilon). \end{aligned}$$

Thus, the zero solution of (1.1) is equistable, and the proof is completed. □

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